Author John J Weber III, PhD | **email** John.Weber@gpc.edu
Updated 29 December 2014 | **Page** 1/2
URL: http://www.jjw3.com/r/uSub.pdf

jjw3.com

1 u Substitution

1.1 Requirements

- Need to know how to differentiate ().
- Need to identify both:
  1. a function to be substituted with u, and
  2. the derivative of the u function.

Potential exceptions are: $u = \sqrt{x}$ and $u = e^x$.

1.2 Possible Substitutions

1. Integrand consists of a composition of functions, $f(g(x))$
   - let $u = g(x)$, and
   - $du = g'(x)\,dx$
   - then $\displaystyle\int f(g(x))\,g'(x)\,dx$ becomes $\displaystyle\int u du$
     - **Example**: $\int xe^{x^2}dx$
       (a) Since there is a composition of e^x and x^2 while x^2 is the 'inside' function, let $u = x^2$
       (b) $du = 2xdx$; rewrite this expression as $dx = \dfrac{du}{2x}$
       (c) substitute: $\int xe^{x^2}dx$ becomes $\displaystyle\int xe^u\frac{du}{2x}$
       (d) simplify to: $\displaystyle\frac{1}{2}\int e^u du$
       (e) integrate: $\dfrac{1}{2}e^u + C$
       (f) substitute back into the original variable: $\int xe^{x^2}dx = \dfrac{1}{2}e^{x^2} + C$
2. Integrand consists of a rational expression, $\dfrac{f(x)}{g(x)}$
   (a) let $u =$ the entire denominator
     - **Example**: $\displaystyle\int \frac{x}{1+x^2}dx$
       i. let $u = 1 + x^2$
       ii. $du = 2xdx$; rewrite this expression as $dx = \dfrac{du}{2x}$
       iii. substitute: $\displaystyle\int \frac{x}{1+x^2}dx$ becomes $\displaystyle\int \frac{x}{u}\frac{du}{2x}$
       iv. simplify to: $\displaystyle\frac{1}{2}\int \frac{1}{u}du$

v. integrate: $\frac{1}{2}\ln|u| + C$

vi. substitute back into the original variable:
$\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln\left|1+x^2\right| + C$

(b) let $u =$ a factor within the denominator

- **Example**: $\int \frac{\cos(x)}{1+\sin^2(x)}dx$

  i. Since there is a composition of $\sin(x)$ and x^2 in the denominator, let $u = \sin(x)$

  ii. $du = \cos(x)dx$; rewrite this expression as $dx = \frac{du}{\cos(x)}$

  iii. substitute: $\int \frac{\cos(x)}{1+\sin^2(x)}dx$ becomes $\int \frac{1}{1+u^2}du$

  iv. no simplification needed

  v. integrate: $\tan^{-1}(u) + C$

  vi. substitute back into the original variable:
  $\int \frac{\cos(x)}{1+\sin^2(x)}dx = \tan^{-1}(\sin(x)) + C$

3. let $u =$ a factor in the numerator when part of du is in denominator

- **Example**: $\int \frac{\tan^{-1}(x)}{1+x^2}dx$

  (a) Since the derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$ which is recognized in the denominator, let $u = \tan^{-1}(x)$

  (b) $du = \frac{1}{1+x^2}dx$; rewrite this expression as $\left(1+x^2\right)dx = du$

  (c) substitute: $\int \frac{\tan^{-1}(x)}{1+x^2}dx$ becomes $\int \frac{u}{1+x^2}\left(1+x^2\right)du$

  (d) simplify to: $\int u du$

  (e) integrate: $\frac{u^2}{2} + C$

  (f) substitute back into the original variable:
  $\int \frac{\tan^{-1}(x)}{1+x^2}dx = \frac{1}{2}\left(\tan^{-1}x\right)^2 + C$

1.3 Verify

Check the following:

- After the simplifying step, the entire integrand **MUST** be in terms of the new variable, u (do not forget the du)
  - Additional substitutions using the let statement may be necessary;
- For definite integrals, the limits of integration **MUST** be converted into values of u.