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1 Trigonometric Substitution

1.1 Requirements

- Pythagorean idenitites:
  1. $\sin^2(x) + \cos^2(x) = 1$
  2. $\tan^2(x) + 1 = \sec^2(x)$
  3. $1 + \cot^2(x) = \csc^2(x)$

The latter two identities can be derived from the first identity.

- Definitions of the trigonometric functions on the right triangle:
  1. $\sin(t) = \dfrac{\text{opposite}}{\text{hypotenuse}}$
  2. $\cos(t) = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
  3. $\tan(t) = \dfrac{\text{opposite}}{\text{adjacent}}$
  4. $\csc(t) = \dfrac{\text{hypotenuse}}{\text{opposite}}$
  5. $\sec(t) = \dfrac{\text{hypotenuse}}{\text{adjacent}}$
  6. $\cot(t) = \dfrac{\text{adjacent}}{\text{opposite}}$

1.2 Suggested Substitutions

1. if integrand contains $\left(a^2 - x^2\right)^{n/2}$, $n \in \mathbb{Z}$,
   let $x = a\sin(\theta)$, where $a > 0$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$
2. if integrand contains $\left(a^2 + x^2\right)^{n/2}$, $n \in \mathbb{Z}$,
   let $x = a\tan(\theta)$, where $a > 0$, $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$
3. if integrand contains $\left(x^2 - a^2\right)^{n/2}$, $n \in \mathbb{Z}$,
   let $x = a\sec(\theta)$, where $a > 0$, $0 \le \theta < \dfrac{\pi}{2} \cup \pi \le \theta < \dfrac{3\pi}{2}$

1.3 Procedure

1. After identifying the correct substituion, find dx
2. Substitute x and dx into the original integrand
3. Simplify the integrand, make sure that the only variable remaining in the integrand is θ
4. Integrate
5. Use a right triangle to convert the result back into terms of the variable x

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1.4 Example

1. $\int x\sqrt{4-x^2}dx$
2. $a^2 = 4 \Rightarrow a = 2$, let $x = 2\sin(\theta)$
3. $dx = 2\cos(\theta)d\theta$
4. $\int x\sqrt{4-x^2}dx$ becomes $\int 2\sin(\theta)\sqrt{4-(2\sin(\theta))^2}\,[2\cos(\theta)]\,d\theta$
5. $4\int \sin(\theta)\cos(\theta)\sqrt{4-4\sin^2(\theta)}d\theta \Rightarrow 4\int \sin(\theta)\cos(\theta)\sqrt{4\left(1-\sin^2(\theta)\right)}d\theta$
   $4\int \sin(\theta)\cos(\theta)\sqrt{4\left(\cos^2(\theta)\right)}d\theta \Rightarrow 4\int \sin(\theta)\cos(\theta)2\cos(\theta)d\theta$
   $8\int \sin(\theta)\cos^2(\theta)d\theta$
6. let $u = \cos(\theta)$, $du = -\sin(\theta)d\theta \Rightarrow d\theta = \frac{du}{-\sin(\theta)}$, so
   $8\int \sin(\theta)\cos^2(\theta)d\theta$ becomes $8\int \sin(\theta)u^2\frac{du}{-\sin(\theta)} = -8\int u^2du \Rightarrow$
   $-8\left(\frac{u^3}{3}\right) + C = -\frac{8}{3}\cos^3(\theta) + C$
7. Use a right triangle to convert the result back into terms of the variable x
   (a) From the let statement: $x = 2\sin(\theta) \Rightarrow \sin(\theta) = \frac{x}{2}$
   (b) Construct the right triangle (shown below)
   (c) By Pythagorean Theorem: adjacent leg $= \sqrt{2^2 - x^2}$
   (d) Using the definition of $\cos(\theta)$ on the right triangle, $\cos(\theta) = \frac{\sqrt{4-x^2}}{2}$
   (e) Thus, $\int x\sqrt{4-x^2}dx = -\frac{8}{3}\cos^3(\theta) + C = -\frac{8}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + C \Rightarrow$
   $-\frac{1}{3}\left(\sqrt{4-x^2}\right)^3 + C$